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The asymptotic behaviour of a starting plume

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A similarity solution is obtained for a model of the turbulent starting plume comprising a steady plume feeding mass, momentum and buoyancy into a vortex ring. Bulk equations representing the time rate of increase of ring momentum and ring buoyancy, together with equations (dependent on broad features of the ring structure) representing the velocity of propagation and time rate of circulation increase are used to determine the motion of the vortex ring. The similarity solution is found to exist only for diffuse distributions of vorticity and buoyancy within the ring. Further, the ratio of ring velocity to plume velocity, which is assumed to be constant, is found to take a value which agrees with that obtained from experimental observations.

1. Introduction

Early theories of atmospheric convection, and in particular cumulus convection, were concerned with two fundamentally different flow types. The first type of flow considered was a steady turbulent plume of small spread angle which entrains ambient fluid across its edge. This type of model has the advantage of considering a steady source of heat flux; however, the small spread angle and lack of top structure limit the model in terms of application to cumulus dynamics. The second type of flow considered was the thermal, a name given by glider pilots to a single heated volume, or bubble, of air which ascends in an unsteady manner. Thermals are known to entrain fluid partly through their sides and top but primarily through the rear. This rising and expanding flow is more easily recognized as being of a similar form to cumulus-type convection but has no continuing source of buoyancy.

Turner (1962) pointed out that improved cloud models should consider certain features of both the above models and that, despite the fact that present plume models and thermal models have a different functional dependence of velocity upon height, the two may be combined to give a similarity solution for a starting plume. In his model the plume feeds both momentum and buoyancy into the cap, or thermal, through its base, the thermal moving at a constant fraction of the mean velocity of the plume. While Turner was able to show the model to be feasible in terms of a similarity solution, he was not able to describe the motion in the thermal beyond saying that it was like that in a spherical vortex. It is proposed to extend Turner's model, by considering the thermal to be a vortex ring of more general type, in order to determine how diffuse the distribution of vorticity in the thermal must be for the thermal to achieve self-similarity and also to compare the theoretical value of the velocity ratio with that observed by Turner.

The equations of this more general type of isolated vortex ring are presented in § 3, recent progress in the theory of vortex rings enabling the vortex motion to be predicted with greater accuracy than has been previously possible. McGregor (1974) has pointed out that the velocity of propagation of a vortex ring depends not only on the ring circulation and ring radius, but also on certain broad features of the vorticity distribution within the ring itself. A survey of elements of vortex-ring structure by Morton (1971) describes the relationships of bulk equations and the possible states of self-similarity. It will be convenient here to make use of Morton's expression for the time rate of change of circulation of the ring, which depends upon the balance obtained between buoyant generation of vorticity and loss of vorticity by diffusion across the axis of symmetry. The third equation required to specify the motion is the bulk equation described by Lamb (1932) which represents the momentum of all fluid carried with the ring.

The construction of the model is outlined in § 4, the vortex ring being assumed to move at a constant fraction of the mean velocity of the plume, which then feeds mass, momentum, vorticity and buoyancy into the thermal through its base. The extra ring-wise vorticity advected from the plume acts to modify the circulation while the advected momentum and buoyancy act to increase the total momentum and buoyancy of the ring, under the assumption that all quantities become effective vortex-ring quantities as soon as they cross the thermal's boundary. The result is a modified set of equations representing the vortex momentum, velocity of propagation and time rate of increase of circulation.

Further progress with the analysis requires the vorticity and buoyancy profiles in the ring to be specified so that the rates of buoyant generation and diffusive loss of circulation may be found. These profiles are presented in § 5, and various associated length scales are calculated. However, the exact forms of the profiles are not critical, as the equations of motion depend only on quantities which are averaged throughout the thermal. Once the turbulent diffusivity has been appropriately defined, asymptotic solutions to the modified equations of motion are found provided that the circulation increases as $(\text{time})^{\frac{1}{2}}$.

Removing all dimensional dependence in § 6 then yields expressions which enable the ratio of thermal to plume velocity to be determined in terms of known quantities and the concentration of vorticity and buoyancy within the vortex. These expressions constitute the solution to the problem, the remaining sections dealing with the thermal entrainment rate, previous experimental results, the present theoretical results and the conclusions.

2. Equations of the steady plume

Similarity solutions for steady turbulent forced plumes were investigated by Morton, Taylor & Turner (1956) using equations representing conservation of mass, momentum and buoyancy. Since then, solutions to this set of equations have been refined by Morton (1959) and Morton & Middleton (1973). These models assume that local variations in buoyancy are small compared with some reference value; that profiles of mean vertical velocity and mean buoyancy are each of similar form at all plume cross-sections; that there exists an entrainment constant relating the rate of inflow to the mean vertical velocity, and that Reynolds number similarity holds for the turbulent fluid within the plume. The above assumptions are fully discussed in the above references and in a review written by Turner (1969) and, although the resultant models are rather idealized, there is evidence to suggest that they give serviceable results. Accordingly, for the proposed starting-plume model, it is sufficient to use the results for an idealized simple plume.

Cylindrical polar co-ordinates (σ, θ, z) with z vertically upwards are used and the plume motion is assumed to be axisymmetric and without swirl. Gaussian profiles, representing observed values across the plume, are chosen to characterize profiles of mean vertical velocity and mean buoyancy and these may be written as

$$\left.\begin{aligned} u(z,\sigma) &= U(z)\exp(-\sigma^2/b^2) \\ \text{and} \quad \beta g(T-T_e) &= \rho_e^{-1}g(\rho_e-\rho) = P(z)\exp(-\sigma^2/b^2). \end{aligned}\right\} \tag{1}$$

As (1) indicate, the buoyancy may be due to either a temperature excess or a density deficiency, with the subscript e representing ambient values and ρ, T and β representing density, temperature and the coefficient of cubical expansion respectively. For simplicity the lateral spread of heat is assumed to be the same as the lateral spread of vertical momentum, the velocity profile and the buoyancy profile then both being characterized by the same radial length scale $b(z)$. Rouse, Yih & Humphreys (1952) have, however, conducted experiments above isolated gas flames in air and found the lateral spread of heat to be greater than that of the vertical momentum by a factor of 1·16. Neglect of this factor should have no qualitative effect on the solution and it is anticipated that the quantitative effect should be well masked by other approximations.

Following Morton *et al.* (1956), the rate of inflow per unit height of ambient fluid into the plume is assumed to be $2\pi b\alpha U$, the quantity α being the entrainment constant. For Gaussian profiles its value is thought to be approximately $\alpha = 0{\cdot}09$ but there is evidence to suggest that α is not the same for all kinds of forced plume (see, for example, Turner 1969; Abraham 1965; Morton 1959). Thus it will be necessary to examine the dependence of the solution on the entrainment constant.

With the above assumptions, equations of conservation of mass, momentum and buoyancy may be written as

$$\frac{d}{dz}(b^2U) = 2\alpha bU, \quad \frac{d}{dz}(b^2U^2) = 2b^2P, \quad \frac{d}{dz}(b^2UP) = 0. \tag{2}$$

As it is intended to couple plume and thermal equations, it is necessary to convert the plume equations to Lagrangian form since the thermal equations will all be derived on a time-dependent basis.

These Lagrangian equations may be derived from (2) by considering the change of variable from z to t given by $d/dz = U^{-1}d/dt$, or may be found by consideration of an element of fluid as it moves up the plume. Ideally, a Lagrangian approach relies upon an element of fluid being able to retain its identity as it moves with the flow. While this is not strictly true in a plume the vertical diffusion is significantly less than the radial diffusion and a Lagrangian representation may be thought to be physically reasonable, but, in any case, the transformation is primarily one of convenience and is not required to describe the flow exactly.

The Lagrangian conservation equations are thus

$$\frac{d}{dt}(b^2U) = 2\alpha bU^2, \quad \frac{d}{dt}(b^2U^2) = 2b^2UP, \quad \frac{d}{dt}(b^2UP) = 0. \tag{3}$$

Let the mass, momentum and buoyancy fluxes be denoted by W, M and F in the form

$$W = b^2U, \quad M = \tfrac{1}{2}b^2U^2, \quad F = \tfrac{1}{2}b^2UP. \tag{4}$$

Solutions to (3) are then given by

$$\left.\begin{aligned} F &= F_0, \quad M = M_0 + 2F_0(t-t_0) \\ \text{and} \qquad W &= 2^{\frac{3}{4}}\, 5^{-\frac{1}{4}}\, \alpha^{\frac{1}{2}} F_0^{-\frac{1}{4}} [M_0 + 2F_0(t-t_0)]^{\frac{5}{4}}, \end{aligned}\right\} \tag{5}$$

where the zero subscripts denote values at the actual plume source. These solutions enable fluxes to be calculated as a function of both the source conditions and the elapsed time interval since the plume fluid element left the source.

One further aspect of the Gaussian velocity profile will be of interest when comparing theory with experiment. The volume flux calculated using the Gaussian velocity profile $U(z)\exp(-\sigma^2/b^2)$ and evaluated over an infinite cross-section is $\pi b^2 U$. If this volume flux is considered as being effective over the more realistic cross-sectional area of πb^2, then the value $U(z)$ must be considered as the average value rather than the maximum value. Thus the maximum velocity $U(z)$ which appears in the analysis will actually represent the average value observed in experiments.

3. Equations of the isolated vortex ring

The cap of the starting plume is considered to have the structure of a vortex ring or thermal. In early work on thermals, bulk properties were thought sufficient to determine the flow characteristics uniquely, but more recent work by McGregor (1974) shows the velocity of propagation to depend not only on the ring circulation but also on certain length scales associated with the vortex structure. McGregor experimented numerically with many different vorticity distributions and determined the mean velocity of propagation of a vortex ring to be most accurately described by an equation of the form

$$\frac{d\bar{z}}{dt} = v = \frac{K}{4\pi\bar{\sigma}}\left[\ln\left(\frac{4\bar{\sigma}}{\bar{a}}\right) - \tfrac{1}{4}\right]. \tag{6}$$

The ring circulation is denoted by K, and the ring radius $\bar{\sigma}$, ring sub-radius $\bar{a}$ and average ring height $\bar{z}$ are defined by the following equations:

$$\left.\begin{aligned} R^2 &= \frac{\iint \sigma^2\omega\, d\sigma\, dz}{\iint \omega\, d\sigma\, dz}, \quad h^2 = \frac{\iint z^2\omega\, d\sigma\, dz}{\iint \omega\, d\sigma\, dz}, \quad \bar{z} = \frac{\iint z\omega\, d\sigma\, dz}{\iint \omega\, d\sigma\, dz}, \\ \bar{\sigma} &= \frac{\iint \sigma\omega\, d\sigma\, dz}{\iint \omega\, d\sigma\, dz}, \quad \bar{a} = \tfrac{1}{4}(R^2-\bar{\sigma}^2)^{\frac{1}{2}} + \tfrac{3}{4}(h^2-\bar{z}^2)^{\frac{1}{2}}. \end{aligned}\right\} \tag{7}$$

Here the integrals are evaluated over the entire region of vorticity ω.

In a survey paper on vortex-ring structure Morton (1971) derived a vorticity conservation equation of the form

$$\frac{dK}{dt} = -2\nu_T \int_{-\infty}^{\infty} \left[\frac{\partial \omega}{\partial \sigma}\right]_{\sigma=0} dz + \beta g \int_{-\infty}^{\infty} [T - T_e]_{\sigma=0}\, dz. \tag{8}$$

The time rate of increase of circulation thus depends upon the balance obtained between diffusion of vorticity across the axis of symmetry and buoyant generation of vorticity resulting from the axial temperature being greater than the ambient value. Here the turbulent diffusivity is represented by ν_T.

Some common limiting types of vortex ring are easily identified by examination of (8). A convective vortex is one in which the vorticity is concentrated primarily in a thin toroidal core, the diffusion of vorticity across the axis of symmetry then being zero. For convective buoyant vortices there is also zero net buoyant generation of vorticity and hence the ring circulation remains constant. These types have been considered by Turner (1957) for both the laminar and the turbulent cases. On the other hand a diffusive vortex has vorticity and buoyancy spread more generally through the migrating fluid and is at a later stage of turbulent decay. According to the relative distributions of vorticity and buoyancy within the vortex the circulation may remain constant as in a weak thermal vortex (see, for example, Morton 1960) or may decrease as in a neutral diffusive vortex. All distributions of vorticity must be zero on the axis of symmetry and have a toroidal maximum at some finite radius, regardless of whether the vortex is convective or diffusive. It is assumed in the following that vorticity and buoyancy are diffused equally by the turbulence, both distributions then having the same spread. The buoyancy profile will, nevertheless, differ in general form from the vorticity profile for vortices which are diffusive, since only a single maximum of the temperature excess will exist and this will be situated on the axis of symmetry. For thin ring vortices the general form will be the same, the temperature excess also exhibiting a toroidal maximum and a zero on the axis of symmetry.

The equation giving the impulse of all the fluid moving with a vortex ring was developed by Lamb (1932, p. 239) and is

$$P_c = \pi \rho K R^2, \tag{9}$$

where ρ is the mean density of the advected fluid. Here all variations in density are assumed small compared with the ambient density and the radial length scale R is defined as in (7). In the case of an isolated buoyant thermal the buoyancy force acts to increase the thermal momentum, the time rate of change of momentum being

$$dP_c/dt = \rho_e F_c, \tag{10}$$

where ρ_e is the ambient fluid density and F_c is the total buoyancy of the thermal.

4. Equations of the modified vortex ring

The model is posed as consisting of a thermal propagating at the head of a steady plume, the general outline of the turbulent flow having the shape of an ice-cream cone. The thermal moves at a constant fraction of the mean velocity

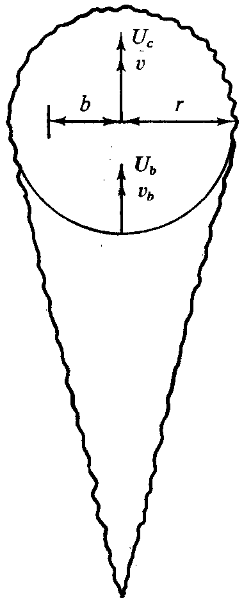


FIGURE 1. Diagram of a starting plume, showing the visible radius r of the thermal, the mean thermal velocity v and the theoretical values of the plume radius b and mean plume velocity U_c at a height equivalent to the thermal centre. The mean velocity v_b of the thermal's base and the mean plume velocity U_b at that height are also shown.

of the plume and the plume thus feeds mass, momentum, circulation and buoyancy into the thermal.

A general idea of the construction of the model is given in figure 1, where r denotes a characteristic radius of the thermal, whose mean vertical velocity is v. This radius must of course be related to the length scales defined in (7) but for the time being it is sufficient to define r as being the average visible radius of the thermal, which is assumed to be spherical. As the thermal volume is increasing with time the average velocity v_b of the thermal base will be somewhat smaller than v. If U_b denotes the mean plume velocity in a steady plume at a height equivalent to that of the base of the thermal, the basic assumption is readily expressed as

$$v_b = A_b U_b, \tag{11}$$

where A_b is a constant whose value is less than one. It follows from (11) that the mean thermal velocity v is proportional to the mean plume velocity U_c at a height equivalent to the thermal centre and so

$$v = AU_c, \tag{12}$$

but, since Turner's measurements were of the cap front velocity v_t and the steady plume velocity U_t at the top of the cap, the third equation

$$v_t = A_t U_t \tag{13}$$

is also required in order to compare theoretical and experimental results. The relationships between A, A_b and A_t may readily be found by considering the plume equations (2) and the rate of growth of the cap radius. If $\alpha' = dr/dz$ is the semi-spread angle of the visible cap edge then these relationships are

$$A_b = A(1-\alpha')^{\frac{4}{3}}, \quad A_t = A(1+\alpha')^{\frac{4}{3}}. \tag{14}$$

Photographs taken by Turner (1962) seem to indicate that the thermal grows in step with the plume, the value of the spread angle of the thermal bearing a constant relation to the plume spread angle. This may readily be expressed as

$$r = Db, \tag{15}$$

where b is the characteristic plume radius and D is a constant.

The Lagrangian solutions (5) give the dependence upon time of the mass, momentum and buoyancy fluxes as a particular element moves up the plume. These values are required at the height of the thermal's base, but the plume fluid element at this height at any particular time is younger than the thermal by a factor of A_b. Thus if t represents the time which the thermal has taken to rise from its source, then the element of plume which has just arrived at the thermal's base must have taken a time $A_b t$ to rise from the same source. All plume quantities evaluated at the thermal's base may therefore be written as a function of the time the thermal has taken to rise by replacing the time variable in the plume quantities by $A_b t$.

Asymptotic solutions for large times are required in order that the flow is able to settle into a similarity state and so the equations of conservation of momentum, buoyancy and mass in the plume at the thermal's base are approximated by

$$\left.\begin{aligned} M_b &= 2F_0 A_b t, \quad F_b = F_0 \\ \text{and} \qquad W_b &= 8\alpha^{\frac{1}{2}} 5^{-\frac{1}{2}} F_0^{\frac{3}{4}} A_b^{\frac{5}{4}} t^{\frac{5}{4}}, \end{aligned}\right\} \tag{16}$$

where as before the subscript b denotes values at the thermal's base and t represents the time of rise of the thermal.

The equations for the thermal of the starting plume must now be modified to allow for the extra buoyancy and momentum supplied by the plume. As a result the buoyancy of the thermal increases with time rather than remaining constant as in the case of an isolated thermal and the asymptotic form for the thermal buoyancy is

$$F_c = \pi F_0 (1-A_b) t. \tag{17}$$

Similarly, the rate of change of thermal momentum has a contribution from the advected plume fluid but there is also a buoyant-generation contribution. The asymptotic thermal momentum is thus

$$P_c = \rho\pi(\tfrac{1}{2}+A_b)(1-A_b) F_0 t^2. \tag{18}$$

Consider next the circulation equation for the cap. In addition to the diffusion term and the generation term for an isolated thermal, there will also be a contribution to the thermal circulation resulting from the assimilation of plume fluid which has ring-wise vorticity. The ring-wise circulation of a plume, evaluated around a contour consisting of a unit length segment up the plume axis together with two radial segments extending to and joining at infinity, is given by the magnitude of the plume velocity U. At the thermal's base this circulation is U_b, and is advected into the thermal with velocity $U_b - v_b$. Thus the rate of change of circulation resulting from the advected plume fluid is $U_b(U_b - v_b)$, which, with the aid of (4) and (16), may be put in the asymptotic form

$$U_b(U_b - v_b) = (5/4\alpha)\, F_0^{\frac{1}{2}}(1 - A_b)\, A_b^{-\frac{1}{2}} t^{-\frac{1}{2}}.$$

The three modified equations representing the motion of the thermal of a starting plume may now be rewritten in the form

$$\left.\begin{aligned} \frac{dK}{dt} &= -2\nu_T \int_{-\infty}^{\infty} \left[\frac{\partial \omega}{\partial \sigma}\right]_{\sigma=0} dz + \beta g \int_{-\infty}^{\infty} [T - T_e]_{\sigma=0}\, dz + (5/4\alpha)\,(1 - A_b)\, A_b^{-\frac{1}{2}} F_0^{\frac{1}{2}} t^{-\frac{1}{2}}, \\ P_c &= \pi\rho K R^2 = \pi\rho(\tfrac{1}{2} + A_b)\,(1 - A_b)\, F_0 t^2, \\ v &= \frac{K}{4\pi\bar{\sigma}}\left[\ln\left(\frac{4\bar{\sigma}}{\bar{a}}\right) - \tfrac{1}{4}\right] = \frac{5^{\frac{1}{2}}}{2\alpha^{\frac{1}{2}}} A^{\frac{3}{4}} F_0^{\frac{1}{4}} t^{-\frac{1}{4}}. \end{aligned}\right\} \tag{19}$$

5. Vorticity and buoyancy distributions in the thermal

In order to proceed further with the analysis, distributions of vorticity and buoyancy are required for the thermal. Gaussian-type distributions are proposed, the distributions of vorticity and buoyancy both containing exponential functions having the same mean σ_0 and same standard deviation L, in accordance with the assumption of equal diffusivities of buoyancy and vorticity made earlier. The distributions for $\sigma > 0$ are thus assumed to be of the form

$$\left.\begin{aligned} \omega &= \frac{K}{r^2} \exp\left[\frac{(z - z_0)^2}{-L^2}\right]\left\{\exp\left[\frac{(\sigma - \sigma_0)^2}{-L^2}\right] - \exp\left[\frac{(\sigma + \sigma_0)^2}{-L^2}\right]\right\} \\ \text{and} \quad \beta g(T - T_e) &= \frac{F_c}{r^3} \exp\left[\frac{(z - z_0)^2}{-L^2}\right]\left\{\exp\left[\frac{(\sigma - \sigma_0)^2}{-L^2}\right] + \exp\left[\frac{(\sigma + \sigma_0)^2}{-L^2}\right]\right\}, \end{aligned}\right\} \tag{20}$$

where K and F_c are the thermal circulation and buoyancy defined earlier.

It must be recognized that the information that the equations of motion (19) require from the distributions (20) is information which is of an averaged rather than a specific nature. This is more easily seen when one considers that the length scales (7) are determined by integrals involving weighted averages of the vorticity distribution, the integrals being evaluated over the entire vorticity field. The terms representing diffusive loss and buoyant generation of vorticity are also average values obtained by integration of the vorticity gradient and temperature excess along the axis of symmetry from the top to the bottom of the thermal. Thus the exact choice of distributions cannot be a critical factor in determining the equations of motion, as the model depends only on certain broad features of these distributions.

Define now a parameter B such that $L = B\sigma_0$. This parameter will then define uniquely the spread (diffuse or concentrated) of vorticity and buoyancy throughout the thermal and in conjunction with the ratio D, defined in (15), will specify

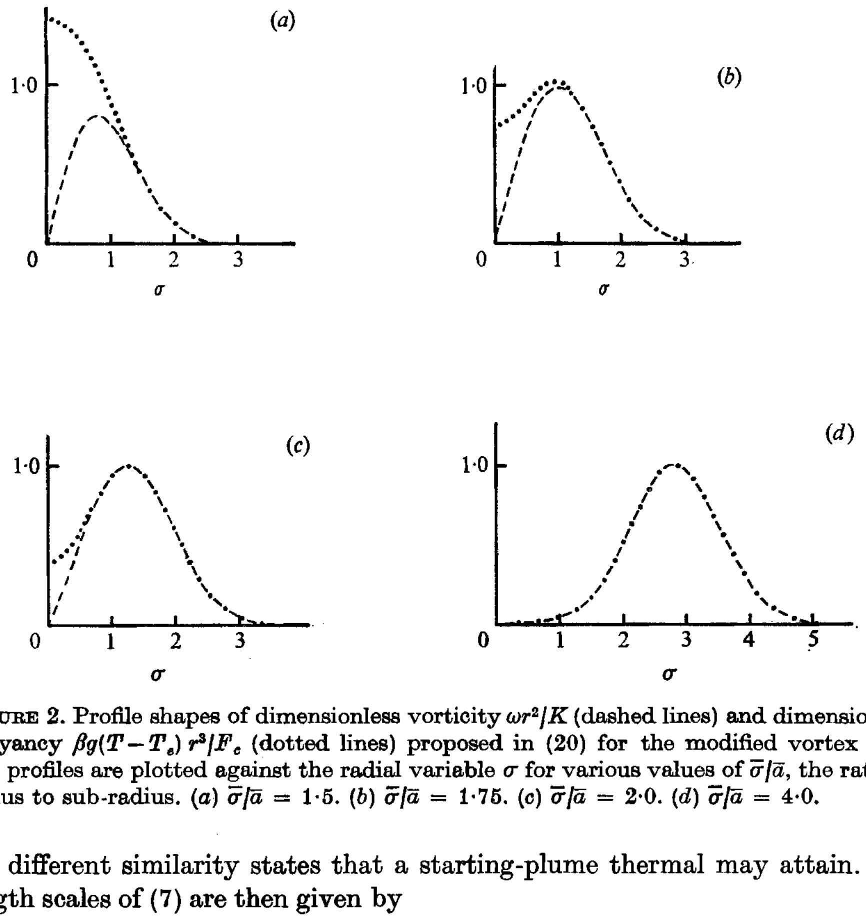


FIGURE 2. Profile shapes of dimensionless vorticity $\omega r^2/K$ (dashed lines) and dimensionless buoyancy $\beta g(T-T_e)\, r^3/F_c$ (dotted lines) proposed in (20) for the modified vortex ring. The profiles are plotted against the radial variable σ for various values of $\overline{\sigma}/\overline{a}$, the ratio of radius to sub-radius. (a) $\overline{\sigma}/\overline{a} = 1{\cdot}5$. (b) $\overline{\sigma}/\overline{a} = 1{\cdot}75$. (c) $\overline{\sigma}/\overline{a} = 2{\cdot}0$. (d) $\overline{\sigma}/\overline{a} = 4{\cdot}0$.

the different similarity states that a starting-plume thermal may attain. The length scales of (7) are then given by

$$\left.\begin{aligned} \overline{\sigma} = \sigma_0 w_1, \quad R^2 = \sigma_0^2 w_2, \quad \overline{z} = z_0, \\ h^2 = z_0^2 + \tfrac{1}{2}B^2\sigma_0^2, \quad \overline{a} = \sigma_0 w_3, \end{aligned}\right\} \tag{21}$$

where the weighting coefficients w_1, w_2 and w_3 are given by

$$\left.\begin{aligned} w_1 &= (\operatorname{erf} B^{-1})^{-1}, \\ w_2 &= 1 + \tfrac{1}{2}B^2 + B\exp(-B^{-2})\,(\pi^{\frac{1}{2}}\operatorname{erf} B^{-1})^{-1}, \\ w_3 &= \tfrac{1}{4}[(w_2 - w_1^2)^{\frac{1}{2}} + 3B2^{-\frac{1}{2}}]. \end{aligned}\right\} \tag{22}$$

In order that the reader may more easily understand how the profile shapes (20) are affected by variation of the radius $\overline{\sigma}$ and sub-radius $\overline{a}$, figure 2 plots the dimensionless vorticity $\omega r^2/K$ and dimensionless buoyancy $\beta g(T-T_e)\, r^3/F_c$ against the radial variable σ for various values of $\overline{\sigma}/\overline{a}$. To simplify comparison, all profiles have been drawn with $L = 1$. Equations (21) and (22) show that the value of B uniquely defines $\overline{\sigma}/\overline{a}$, but it is thought that the ratio $\overline{\sigma}/\overline{a}$ of radius to sub-radius is an easier parameter to comprehend physically and, in any case, $\overline{\sigma}/\overline{a}$ represents a quantity more directly pertinent to the equations of motion. It must be emphasized once more, however, that the equations of motion depend only on certain mean quantities associated with the profile shapes and not on the exact shapes themselves.

It is necessary to define the characteristic thermal radius r in terms of the ring radius $\overline{\sigma}$ and sub-radius $\overline{a}$. It is thought sufficient to approximate this visible thermal radius by

$$r = \overline{\sigma}+\overline{a} \tag{23}$$

since this expression is the simplest available; however this approximation may be assessed only after the solutions have been found. Comparison with (15) shows the ratio of the visible thermal radius $\overline{\sigma}+\overline{a}$ to the characteristic plume radius b to be given by the ratio of spread angles, $\alpha'/\frac{6}{5}\alpha$, and thus the similarity constant D is given by $D = 5\alpha'/6\alpha$.

Use of (20) and (21) also enables the integrals representing diffusive loss and buoyant generation of circulation to be evaluated for the circulation equation (19). Further manipulation of (21) and the asymptotic plume equations enables the quantities R^2, r^2 and $\overline{\sigma}$ to be represented as

$$R^2 = \frac{w_2}{(w_1+w_3)^2}\frac{16\alpha D^2}{5}F_0^{\frac{1}{2}}A^{\frac{3}{2}}t^{\frac{3}{2}},$$

$$r^2 = \tfrac{16}{5}\alpha D^2 F_0^{\frac{1}{2}}A^{\frac{3}{2}}t^{\frac{3}{2}}$$

and

$$\overline{\sigma} = \frac{w_1}{(w_1+w_3)}\frac{4\alpha^{\frac{1}{2}}D}{5^{\frac{1}{2}}}F_0^{\frac{1}{4}}A^{\frac{3}{4}}t^{\frac{3}{4}}.$$

Equations (19) may now be written in terms of the circulation K in the form

$$\left.\begin{aligned} \frac{dK}{dt} &= \frac{-\nu_T K}{r^2}A_1+\frac{F_0 t}{r^2}A_2+F_0^{\frac{1}{2}}t^{-\frac{1}{2}}A_3, \\ K &= \frac{5(1-A_b)(\frac{1}{2}+A_b)(w_1+w_3)^2}{16\alpha A^{\frac{3}{2}}D^2 w_2}F_0^{\frac{1}{2}}t^{\frac{1}{2}} \\ \text{and} \qquad K &= \frac{8\pi A^{\frac{3}{2}}Dw_1}{[\ln(4w_1/w_3)-\frac{1}{4}](w_1+w_3)}F_0^{\frac{1}{2}}t^{\frac{1}{2}}, \end{aligned}\right\} \tag{24}$$

where the coefficients A_1, A_2 and A_3 are given by

$$\left.\begin{aligned} A_1 &= 8\pi^{\frac{1}{2}}B^{-1}\exp(-B^{-2}), \\ A_2 &= 2\pi^{\frac{3}{2}}B(1-A_b)(w_1+w_3)^{-1}\exp(-B^{-2}) \\ \text{and} \qquad A_3 &= 1{\cdot}25\alpha^{-1}(1-A_b)A_b^{-\frac{1}{2}}. \end{aligned}\right\} \tag{25}$$

It remains to determine the form of the turbulent diffusivity ν_T, which has dimensions of (length)^2 (time)$^{-1}$. It is appropriate to scale this quantity according to the characteristic thermal velocity scale v and the thermal length scale $\overline{a}$, the sub-radius, which represents the spread of vorticity and buoyancy about the mean radius $\overline{\sigma}$. Thus

$$\nu_T = E\overline{a}v, \tag{26}$$

where E is a dimensionless eddy-diffusivity constant dependent on the similarity state. For large times ν_T approaches the asymptotic value

$$\nu_T = 2EA^{\frac{3}{2}}Dw_3(w_1+w_3)^{-1}F_0^{\frac{1}{2}}t^{\frac{1}{2}}. \tag{27}$$

For similarity, (24) must hold for large times and this is only possible if the circulation increases as (time)$^{\frac{1}{2}}$. Thus the asymptotic starting-plume vortex requires an increase in vorticity with time rather than the decrease which is normally expected for an isolated vortex.

6. Scaling

In order to remove the time dependence and the dimensional quantities from (24) and (27), the simple transformations

$$t = M_0 F_0^{-1}\tau, \quad K = M_0^{\frac{1}{2}}\kappa \tag{28}$$

must be applied. The time rate of increase in circulation may then be represented as

$$\kappa = H\tau^{\frac{1}{2}}, \tag{29}$$

the value of H being constant for a similarity state.

Choice of M_0 and F_0 as a base for transformation to dimensionless form is rather arbitrary, there being three choices of base available from the source values M_0, F_0 and W_0 of the momentum flux, buoyancy flux and mass flux. If the lateral spread of momentum is taken as being the same as the lateral spread of buoyancy, as assumed earlier, then reference to Morton & Middleton (1973) shows the source quantities to be related by

$$M_0 = \tfrac{1}{2}(5/2\alpha)^{\frac{2}{5}} F_0^{\frac{2}{5}} W_0^{\frac{4}{5}}$$

for the case of a simple plume. Thus it does not matter which transformation base is used, and the base (M_0, F_0) is chosen for convenience.

Substitution of (28) and (29) into (24) and (27) enables all time-dependent and other dimensional quantities to be removed and the resultant equation representing the increase in circulation is

$$1 = \frac{-5A_1 E w_3}{4\alpha D(w_1+w_3)} + \frac{5\pi(1-A_b)A_2}{8\alpha D^2 A^{\frac{3}{2}} H} + \frac{5A_3}{8\alpha D^2 A^{\frac{3}{2}} H}. \tag{30}$$

This equation has been normalized with respect to H, the dimensionless rate of increase in circulation, in order that the relative magnitudes of the terms representing buoyant generation, advection and diffusion of circulation may be more readily compared.

The transformed equations of vortex momentum and vortex propagation give the dimensionless rate of circulation increase as

$$H = \frac{8\pi A^{\frac{3}{2}} D}{[\ln(4w_1/w_3) - \frac{1}{4}]} \frac{w_1}{(w_1+w_3)} \tag{31}$$

and

$$H = \frac{5(1-A_b)(\frac{1}{2}+A_b)(w_1+w_3)^2}{16\alpha A^{\frac{3}{2}} D^2 w_2}. \tag{32}$$

Equations (30)–(32) constitute the solution to the problem in terms of three equations and six unknown quantities. This solution is found and discussed in the section dealing with theoretical results (§ 9).

7. The thermal entrainment rate

It will also be informative to calculate the fraction of fluid drawn into the thermal from the plume and thus to find the fraction of ambient fluid entrained directly into the thermal. If V_p denotes the volume of plume fluid in the thermal, then the rate of advection of plume fluid into the cap is given by

$$dV_p/dt = \pi W_b(1-A_b),$$

the factor $1-A_b$ arising from the relative motion of the thermal's base and the plume at that height. The rate of change of cap volume V, however, is given by

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(Db)^2 \frac{dr}{d\bar{z}}\frac{d\bar{z}}{dt},$$

where $dr/d\bar{z}$ is the visible angle of spread α', $d\bar{z}/dt = v$ is the mean velocity of propagation of the thermal and use has been made of (15). A little manipulation shows the product $b^2 v$ evaluated at a height equivalent to the thermal centre to be proportional to the plume mass flux W_c evaluated at this same height. Thus

$$dV/dt = 4\pi D^2 \alpha' A W_c,$$

and the amount of fluid in the thermal which has come directly from the plume is given by

$$\frac{dV_p}{dV} = \frac{(1-A_b)}{4D^2\alpha' A}\left(\frac{A_b}{A}\right)^{\frac{5}{4}}. \tag{33}$$

8. Previous experimental results

The value of B determines A_1, A_2 and A_3 and also the weighting coefficients w_1, w_2 and w_3. Thus (30)–(33) involve six independent unknowns: A, α', α, B, H and E.

The ratio A of cap to plume velocity is one of the easier quantities to measure experimentally and Turner (1962) was able to obtain results for this ratio from his work. Salt solutions of different density were used for 39 runs, the velocity changing by a factor of three as the density changed. It was found convenient to measure the ratio of the velocity of the thermal front to the mean velocity of a filament of dye at the plume axis in the steady flow behind, these measurements being taken at the same height. This was achieved by following first the front and then a filament of dye in the steady flow behind and timing the motion between fixed marks with a stop watch. In this way Turner found the velocity ratio, as defined by (13), to be

$$A_t = 0{\cdot}61 \pm 0{\cdot}05.$$

The half-angle of spread α' of the visible edge of the cap was also measured for a variety of flow rates and, from 18 runs, was found to be

$$\alpha' = 0{\cdot}18 \pm 0{\cdot}03.$$

The plume entrainment constant α, which is a direct measure of the semi-angle of spread $\frac{6}{5}\alpha$ of a simple plume having, in this case, Gaussian profiles of mean vertical velocity and buoyancy, has been evaluated experimentally by many workers. Morton *et al.* (1956) calculated, from their own experimental results and from profiles plotted by Schmidt (1941) and Rouse *et al.* (1952), values of α pertaining to plume flows of 0·093, 0·125, and 0·085 respectively. It was later argued by Morton (1959) that a value of α most representative of the basic turbulent mixing processes should be found from experimental results for jets. Results found by Reichardt (1942), Squire (1950), Kuethe (1935) and Ruden (1933) indicate values of α of 0·081, 0·084, 0·079 and 0·083 respectively. More recent experiments by Ricou & Spalding (1961) suggested that the entrainment rate for buoyant plumes was somewhat greater than that for jets but found a comparatively low value of $\alpha = 0{\cdot}057$ for the jet.

The above results indicate that α most probably lies between 0·08 and 0·12 and similarity solutions corresponding to values of α in this range will be found in order to examine the dependence of the solutions on α. It would seem, however, that the most likely value of α is approximately 0·09, and this has been used by Turner (1962).

Three quantities remain, none of which may easily be found experimentally. Of these, B is a parameter which defines $\overline{\sigma}/\overline{a}$ uniquely, and thus is a direct measure of the spread of vorticity within the thermal. Equations (21) and (22) show that for highly diffusive vortices $\overline{\sigma}/\overline{a}$ takes the value 1·37, but may approach values of order 5 for thin ring vortices of common smoke-ring type. Computationally B is a far easier variable to use than $\overline{\sigma}/\overline{a}$, while the latter is easier to interpret physically. Thus B appears explicitly in the A_t and w_i, but results are plotted against values of $\overline{\sigma}/\overline{a}$ rather than B.

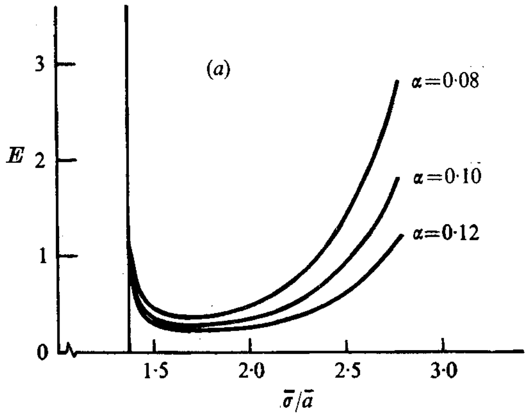
Non-dimensional values of the time rate of increase in circulation H and the turbulent diffusivity constant E would appear to be extremely difficult to measure experimentally and to the author's knowledge no theoretical values have been calculated.

9. Theoretical results

Solution of the equations may readily be achieved by eliminating H from (31) and (32). A choice of values for α' and α then allows the velocity ratio A_t to be plotted against $\overline{\sigma}/\overline{a}$. Appropriate values may then be substituted into either (31) or (32) to determine H, and then into (30) to determine the final unknown quantity E.

In order to interpret the results, however, it is preferable first to consider a plot of E *vs.* $\overline{\sigma}/\overline{a}$ for various values of α and α'. For the similarity solution being sought, the turbulent diffusivity constant E is expected to be a universal constant over the region of validity of the model. Examination of figure 3 shows E to be approximately constant over the experimental range of α' and for various α in the region $1{\cdot}5 < \overline{\sigma}/\overline{a} < 2{\cdot}0$, with the average value of the minima of E occurring at approximately $\overline{\sigma}/\overline{a} = 1{\cdot}75$. This value is consistent with the assumption made earlier that the vorticity in the thermal is diffuse, and agrees with intuitive ideas that a turbulent flow which has obtained a state of self-similarity must have existed for a sufficiently long time that turbulent diffusion dominates the flow.

A point of interest arising from the model is the form that the eddy diffusivity is assumed to take. In order to agree with the similarity solution the diffusivity must increase as $t^{\frac{1}{2}}$. Consequently it would have been sufficient to have assumed the diffusivity to be proportional to the product of any characteristic thermal length scale and any characteristic velocity scale or, alternatively, to be proportional to the ring circulation. The actual choice made for the eddy diffusivity leads to possible similarity solutions in the range $1{\cdot}5 < \overline{\sigma}/\overline{a} < 2{\cdot}0$ and it seems



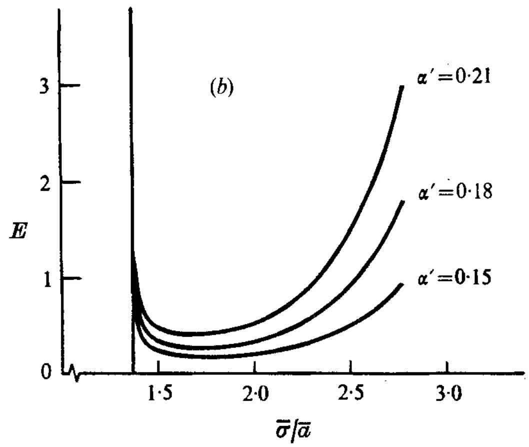


FIGURE 3. Values of the turbulent diffusivity constant E plotted against the various vorticity distributions specified by $\bar{\sigma}/\bar{a}$ for different values of the thermal semi-spread angle α' and the plume entrainment constant α. (a) $\alpha' = 0{\cdot}18$. (b) $\alpha = 0{\cdot}10$.

unlikely that choice of a different scaling would have made much difference to this range, all length scales being proportional as are all velocity scales in a similarity solution of this type.

It can be seen from figure 4 that the velocity ratio A_t is not very sensitive to changes in $\bar{\sigma}/\bar{a}$, the change in A_t amounting to approximately 2 % over the region $1{\cdot}5 < \bar{\sigma}/\bar{a} < 2{\cdot}0$. Thus the choice of $\bar{\sigma}/\bar{a} = 1{\cdot}75$ would seem to be the most representative and realistic estimate of the vorticity concentration which exists in the vortex ring of a starting plume which has achieved similarity.

In order to examine the balance between vorticity production and loss through the cap, the terms in the circulation equation (30) have been plotted as a function of $\bar{\sigma}/\bar{a}$ in figure 5. It may readily be noted that, while the diffusive term is larger in magnitude than any other individual term, the sum of the terms representing

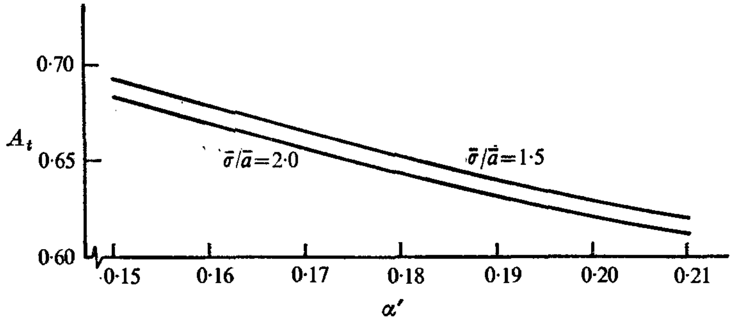


FIGURE 4. Showing the velocity ratio A_t plotted against the thermal semi-spread angle α' for different values of the vorticity distribution $\bar{\sigma}/\bar{a}$. $\alpha = 0·10$.

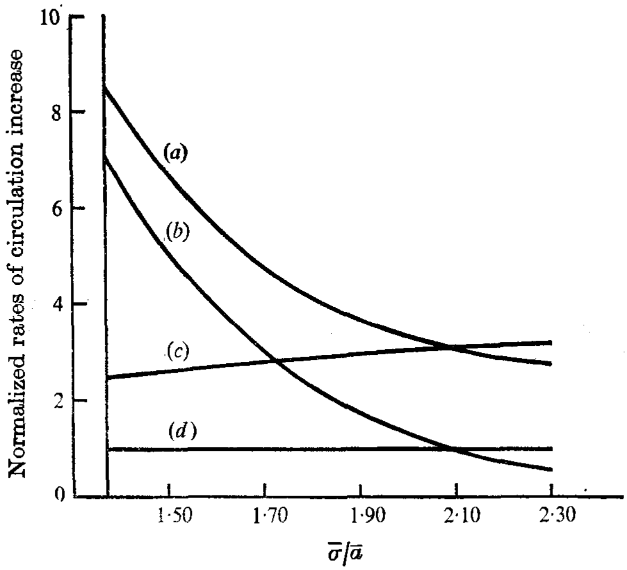


FIGURE 5. Relative magnitudes of terms in (30) representing (*a*) diffusive loss, (*b*) buoyant generation, (*c*) advection and (*d*) resultant rate of increase in circulation, normalized with respect to the dimensionless rate of increase in circulation H. $\alpha = 0·10$, $\alpha' = 0·18$.

vorticity generation (due to buoyancy) and vorticity advection (from the plume) gives a resultant increase in circulation with time. This contrasts with the usual situation found in diffuse vortex rings, where the circulation decreases with time as vorticity diffuses across the axis of symmetry.

The dependence of the non-dimensional rate of increase of circulation on α, α' and $\bar{\sigma}/\bar{a}$ is shown in figure 6. Over the region of interest H is fairly insensitive to changes and, generally speaking, retains a value of about $4\frac{1}{2}$. This insensitivity is probably due to the assumption of equal spread of vorticity and buoyancy, a diffuse vortex tending to gain more vorticity by buoyant generation and lose more by diffusion than a more concentrated vortex ring.

Figure 7 shows the velocity ratio A_t plotted against the visible spread α' of the cap and the plume spread angle $\frac{6}{5}\alpha$, the rectangle denoting the experimental

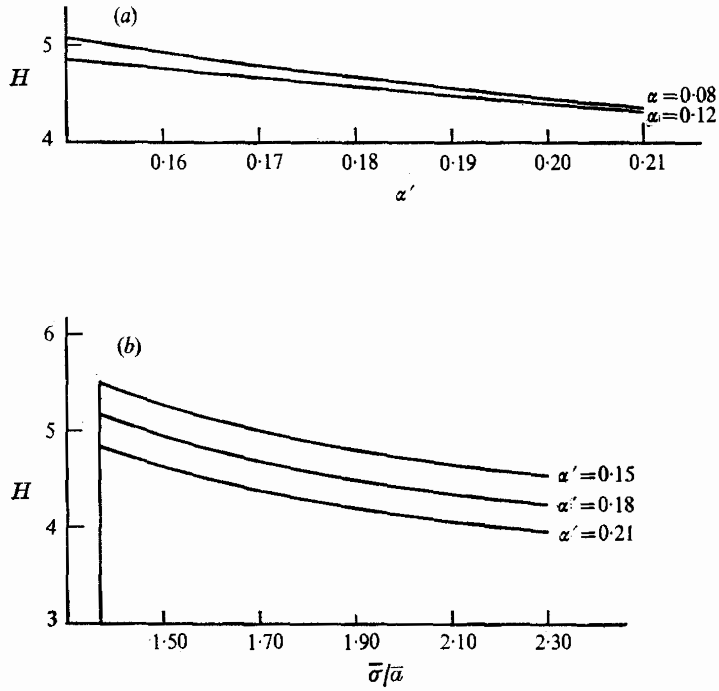


FIGURE 6. The dimensionless rate of circulation increase H, plotted as a function of the thermal semi-spread angle α', the vorticity distribution $\bar{\sigma}/\bar{a}$ and the plume entrainment constant α. (a) $\bar{\sigma}/\bar{a} = 1{\cdot}75$. (b) $\alpha = 0{\cdot}10$.

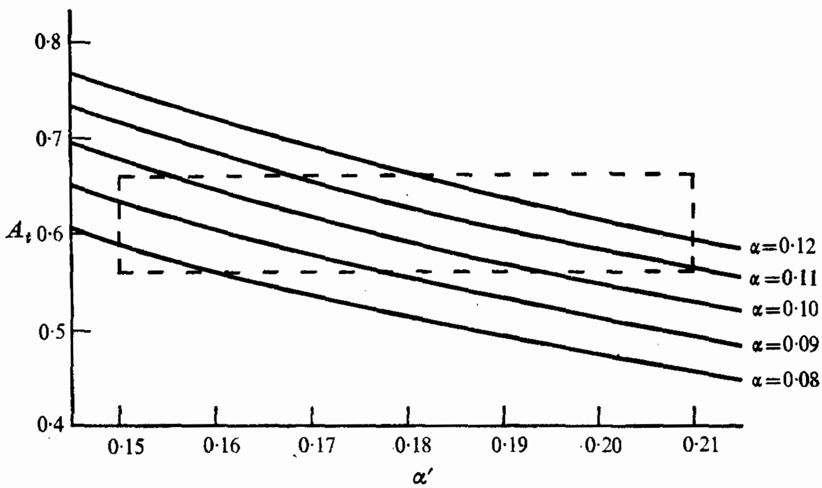


FIGURE 7. The ratio A_t of cap front velocity to steady plume velocity measured at the same height, plotted as a function of the thermal semi-spread angle α' and the plume entrainment constant α. The rectangle represents Turner's (1962) experimental results. $\bar{\sigma}/\bar{a} = 1{\cdot}75$.

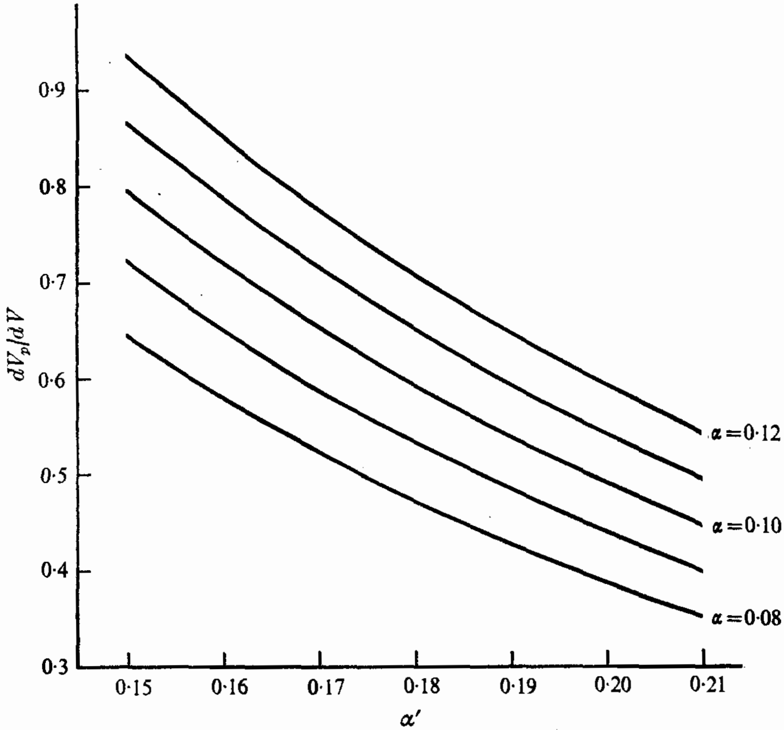


FIGURE 8. The fraction dV_p/dV of thermal fluid obtained directly from the plume plotted as a function of the thermal semi-spread angle α' and the plume entrainment constant α. $\overline{\sigma}/\overline{a} = 1{\cdot}75$.

variation in A_t and α' found by Turner (1962). Theoretical values of A_t for $\alpha = 0{\cdot}09$, whilst being slightly low, are in reasonable agreement with the experimental results, and it is interesting to note that agreement is generally good over the whole range of values of α.

The fraction of thermal fluid which has come directly from the plume is given as a function of α and α' in figure 8. For $\alpha = 0{\cdot}09$ and in the experimental range of α', the fraction varies from 0·4 to about 0·7. This implies that about half the thermal fluid is entrained directly from the ambient fluid. This calculation should not be considered as being particularly accurate, as the choice of approximating the visible radius r by $\overline{\sigma}+\overline{a}$ may well be inexact and any error in radius must of course yield a highly magnified error in volume. The reasonable agreement between theoretical and experimental values of A_t would seem, however, to support this choice of definition of the visible radius, further support being lent by the fact that the alternative choice σ_0+L takes a value which is only 7 % greater for $\overline{\sigma}/\overline{a} = 1{\cdot}75$.

It has been assumed that the lateral spread of momentum is the same as the lateral spread of buoyancy in the plume. Consider now the effect of assuming a greater lateral spread of buoyancy as is actually observed in plumes. This greater spread must result in an increase in the buoyancy flux advected by the plume and hence tend to increase the thermal momentum, circulation and velocity slightly above those values plotted.

Top-hat profiles, representing constant mean values of buoyancy and velocity, have sometimes been used in modelling plumes. For this model, however, it is essential that Gaussian profiles (observed experimentally) are used, as top-hat profiles give a much exaggerated value for the buoyancy flux at any height. While this is unimportant for models of a steady plume alone it becomes of major importance in a starting-plume model, where the buoyancy flux in the plume is advected into the thermal, the total buoyancy of which drives the vortex flow.

10. Conclusions

A similarity solution has been obtained for the vortex ring which propagates at the head of a starting plume, the solution being obtained for concentrations of vorticity within the ring having a ratio of radius to sub-radius of approximately $\bar{\sigma}/\bar{a} = 1{\cdot}75$. This value compares favourably with $\bar{\sigma}/\bar{a} = 1{\cdot}5$, the ratio which characterizes the slightly more diffuse spherical vortex which Turner chose for his model. Theoretical values of the ratio of the vortex ring velocity to the plume velocity were calculated for $\bar{\sigma}/\bar{a} = 1{\cdot}75$, the velocity ratio being dependent on the half-angle of spread $\frac{6}{5}\alpha$ of the plume and the half-angle of spread α' of the thermal. Within the experimental ranges of α and α' the calculated values of this velocity ratio were found to be consistent with those measured by Turner.

For $\bar{\sigma}/\bar{a} = 1{\cdot}75$ the magnitudes of the rate of increase in total circulation, the loss of circulation due to diffusion, the gain of circulation due to buoyant generation of vorticity and the gain due to advection from the plume are found to be in the approximate ratio 2:8:5:5. It is interesting to note that the magnitudes of each of the two terms representing circulation production are about the same, and that the introduction of ring-wise vorticity from the plume into the cap must therefore be comparable to the buoyant generation of vorticity. Since either of these terms is greater than the total rate of circulation increase, it is evident that neglect of the term representing advection from the plume will result in the total circulation decreasing with time.

This point is of considerable interest if cumulus towers are thought of as being vortical flows. While it is not being suggested that a cumulus cloud tower ever reaches dynamical similarity, the ratio of height to source width being generally too small, it seems likely that a tower will have much the same character as the thermal of a starting plume. In this situation the balance of vorticity production and loss must have a profound influence on the tower development. If the only source of vorticity, and hence circulation, in the tower is due to buoyancy then the diffusive loss may be greater than the buoyant gain, resulting in a net decrease in circulation with time. The vorticity in the tower will then become more diffuse and the tower will finally dissipate or be enveloped by other towers. If, on the other hand, the net circulation increases with time because of the added influence of ring-wise vorticity from below, then the tower will continue to propagate upwards with progressively increasing strength.

While the actual flow in a cloud tower is complicated by many other factors such as the relative values of the lapse rates and rate of release of latent heat, this balance of vorticity production and loss must play a major role in the tower dynamics.

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